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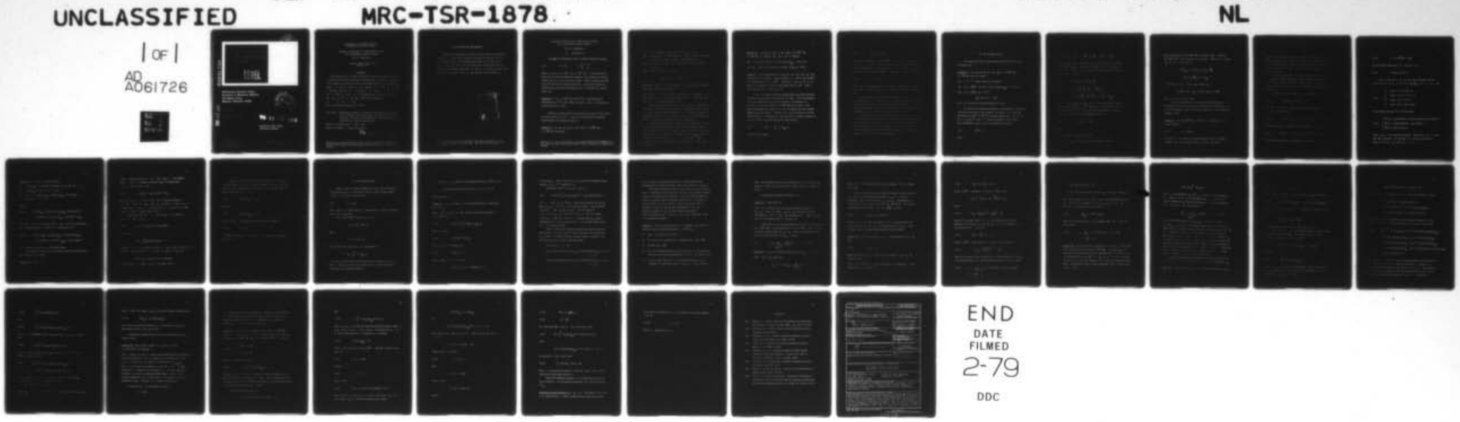

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PERIODIC SOLUTIONS OF A HAMILTONIAN SYSTEM ON A PRESCRIBED ENERGY SURFACE

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PERIODIC SOLUTIONS OF A HAMILTONIAN SYSTEM ON A PRESCRIBED ENERGY SURFACE

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ABSTRACT

The existence of periodic solutions having prescribed energy for a Hamiltonian system of ordinary differential equations is studied. It is shown in particular that if the Hamiltonian is of classical type, i.e., $H(p,q) = K(p,q) + V(q)$ where $p, q \in \mathbb{R}^n$, and K and V satisfy $K(0,q) = 0$, $p \circ K_p(p,q) > 0$, $K(p,q) \to \infty$ as $|p| \to \infty$, $D = \{q \in \mathbb{R}^n | 0 \le V(q) < 1\}$ is diffeomorphic to the unit ball in $\mathbb{R}^n$ and $V_q \neq 0$ on ∂D, then Hamilton's equations,

$$(*) \qquad \dot{p} = -H_q \,, \quad \dot{q} = H_p \,,$$

have a periodic solution on $H^{-1}(1)$.

Key Words: Hamiltonian system of ordinary differential equations, energy surface, periodic orbit, critical point, action integral, canonical transformation, Lagrange multipliers, index theory, equivariant mapping

Subject Classification: 34C15, 34C25

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SIGNIFICANCE AND EXPLANATION

Hamilton's equations are basic in the study of theoretical mechanics. A particular class of motions of interest for (*) are periodic ones. For Hamiltonians which are of the form $H(p,q) = K(p,q) + V(q)$, we give sufficient conditions for the kinetic and potential energies K and V to satisfy so that (*) possesses a periodic orbit on a prescribed energy surface.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

Periodic solutions of a Hamiltonian system on a prescribed energy surface

Paul H. Rabinowitz

§1. Introduction

Consider the Hamiltonian system of ordinary differential equation

$$(1.1)\qquad \dot{z} = \mathcal{J} H_z \ , \qquad \mathcal{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

where $z = (p, q)$, $p, q \in \mathbb{R}^n$, and $H : \mathbb{R}^{2n} \to \mathbb{R}$. A basic question of interest in the study of Hamiltonian mechanics is the existence of periodic solutions of (1.1) on a given energy surface. Simple examples show that there need not exist any periodic orbits [1]. Some recent results of a positive nature were proved simultaneously by A. Weinstein [2] and the author [3]:

<u>Theorem 1.2</u>: Let $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ with $H^{-1}(1)$ a manifold which is diffeomorphic to S^{2n-1} under radial projection. Then (1.1) possesses a periodic solution on $H^{-1}(1)$.

Weinstein actually obtained the special case in which $H^{-1}(1)$ bounds a convex region and $H \in C^2$. A key step in his proof is the following theorem which is of independent interest:

<u>Theorem 1.3</u>: Let $H(p,q) = K(p,q) + V(q)$ where $K \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, $V \in C^2(\mathbb{R}^n, \mathbb{R})$ and satisfy:

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(V_1) If $D = \{q \in \mathbb{R}^n \mid 0 \le V(q) < 1\}$, then $V_q \neq 0$ on ∂D.

(V_2) There exists a C^2 diffeomorphism ψ of $\mathbb{R}^n$ to $\mathbb{R}^n$ such that the open unit ball, $B_1(0)$, in $\mathbb{R}^n$ is diffeomorphic to D.

(K_3) For each $q \in \overline{D}$, $K(0,q) = 0$ and $K(p,q)$ is even and strictly convex in p.

(K_2) For fixed $q \in \overline{D}$ and $p \in S^{n-1}$,

$$\lim_{\alpha \to \infty} K(\alpha p, q) > 1 - V(q) .$$

Then there exists a $T > 0$ and a solution $z = (p,q)$ of (1.1) such that $q(0)$, $q(T) \in \partial D$ and $q(t) \in D$ for $0 < t < T$.

Since $H(z(t)) \equiv 1$ for this solution, $p(0) = p(T) = 0$. Extending p as an odd function and q as an even function about 0 and T and using (K_3) then gives a $2T$ periodic solution of (1.1).

Weinstein's proof of Theorem 1.3 is based in part on earlier work of Seifert [4] on the special case of $K(p,q) = \sum a_{ij}(q) p_i p_j$ where $(a_{ij}(q))$ is a positive definite matrix uniformly for $q \in \overline{D}$. Roughly speaking, the solutions of Theorem 1.3 are determined as geodesics for the Riemannian metric K of Seifert or the Finsler metric K of Weinstein.

A rather different procedure was used to prove Theorem 1.2 in [3]. (See also [5] for another proof). In this paper we will show how the method introduced in [3] can be applied to treat (1.1) under a weakened version of (K_3). To state it, for $a, b \in \mathbb{R}^n$, let $(a,b)_{\mathbb{R}^n}$ denote the usual $\mathbb{R}^n$ inner product of a and b. If $\varphi : \mathbb{R}^\alpha \times \mathbb{R}^\beta \to \mathbb{R}^\gamma$, $\varphi = \varphi(x,y)$, then φ_x denotes the Frechet derivative of φ with respect to x, etc.

Theorem 1.4: Let $H(p,q) = K(p,q) + V(q)$ where $K \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, $V \in C^1(\mathbb{R}^n, \mathbb{R})$, V satisfies $(V_1) - (V_2)$, and K satisfies

(K_1) For $q \in \overline{D}$, $K(0,q) = 0$ and $(p, K_p(p,q))_{\mathbb{R}^n} > 0$ for $p \neq 0$,

and (K_2). Then (1.1) possesses a periodic solution on $H^{-1}(1)$.

Remark 1.5: It is straightforward to verify that $(K_1) - (K_2)$ and $(V_1) - (V_2)$ imply that $H^{-1}(1)$ contains a compact manifold, $\mathfrak{m}$, which is the boundary of a neighborhood Θ of 0 in $\mathbb{R}^{2n}$. Moreover if $(p,q) \in \Theta$ and $q \in D$, then $\Theta_q = \{p \in \mathbb{R}^n \mid (p,q) \in \Theta\}$ is a star-shaped region in $\mathbb{R}^n$. Also if $q \in D$ and $H(p,q) = 1$, then $(p,q) \in \mathfrak{m}$.

In §2, by making a canonical transformation, the proof of Theorem 1.4 will be reduced to the special case of $D = B_1(0)$. This transformation reduces the smoothness of H by one derivative, but fortunately if $D = B_1(0)$, H need only satisfy $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ for our proof. Some further modifications are made to H in §2 converting it to a more suitable Hamiltonian for our methods. Finally in §3 we prove Theorem 1.4 for the modified problem. Following [3], the basic idea is to obtain a solution of (1.1) on $H^{-1}(1)$ as a critical point of the action integral

$$A(z) = \int_0^{2\pi} (p, \dot{q})_{\mathbb{R}^n} \, dt \tag{1.6}$$

subject to the constraint

$$(1.7) \qquad \frac{1}{2\pi} \int_0^{2\pi} H(z)\,dt = 1 .$$

Here $z(t) = (p(t), q(t)) \in E \equiv (W^{1,2}(S^1))^{2n}$, i.e. E is the Hilbert space of $2n$ tuples of 2π periodic functions which together with their first derivatives are square integrable. Any critical point z of A subject to (1.7) and having a non-zero Lagrange multiplier λ will be a 2π periodic solution of

$$(1.8) \qquad \dot{z} = \lambda \mathcal{J} H_z .$$

Since (1.8) is a Hamiltonian system, $H(z(t)) \equiv$ constant so (1.7) shows $z(t) \in H^{-1}(1)$. After rescaling t , z becomes a $2\pi\lambda$ periodic solution of (1.1) .

We do not know a direct way to find critical points of A in E under the constraint (1.7) . Therefore we use the approximation procedure of [3] . This involves restricting A to finite dimensional submanifolds of (1.7) and finding critical points of the approximate problem together with strong enough estimates to pass to a limit and find a solution of (1.8).

An interesting open question in the setting of Theorems 1.2 , 1.3 , or 1.4 is whether stronger statements can be made about the number of distinct periodic orbits on level sets of H . E.g. if H consists of a positive definite quadratic part + higher order terms at $z = 0$, a theorem of Weinstein ([1] or [6]) asserts the existence of at least n distinct periodic solutions on $H^{-1}(c)$ for all sufficiently small $c > 0$.

§2. The modified problem

We begin this section by reducing the proof of Theorem 1.4 to the following result:

Theorem 2.1: Let $H(p,q) = K(p,q) + V(q)$ where $K \in C^1(\mathbb{R}^{2n}, \mathbb{R})$, $V \in C^1(\mathbb{R}^n, \mathbb{R})$ and K, V satisfy

(V_1') $0 \leq V < 1$ in $B_1(0)$ and $V_q \neq 0$ and $\partial B_1(0)$

(K_1') For $q \in \overline{B_1(0)}$, $K(0,q) = 0$ and $(p, K_p(p,q))_{\mathbb{R}^n} > 0$ if $p \neq 0$

(K_2') For $q \in \overline{B_1(0)}$ and $p \in S^{n-1}$,

$$\lim_{\alpha \to \infty} K(\alpha p, q) > 1 - V(q) \quad .$$

Then (1.1) possesses a periodic solution on $H^{-1}(1)$.

To carry out the reduction of Theorem 1.4 to Theorem 2.1, suppose the hypotheses of Theorem 1.4 are satisfied. We extend ψ to a canonical transformation of $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ in a standard fashion [7]. Set $q = \psi(Q)$ and $p = (\psi_Q(Q))^{-T} P$ where G^T denotes the transpose of the matrix G. The transformation $(P,Q) \to (p,q)$ is canonical if and only if

$$(2.2) \qquad \mathcal{L}^T \mathcal{J} \mathcal{L} = \mathcal{J}$$

where

$$\mathfrak{L} = \begin{pmatrix} G & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \quad G = \frac{\partial q}{\partial Q}, \quad \mathcal{B} = \frac{\partial q}{\partial P}, \quad \mathcal{C} = \frac{\partial p}{\partial Q}, \quad \mathcal{D} = \frac{\partial p}{\partial P} .$$

Due to our choice of p, this will be the case if and only if $\psi_Q(Q)^T \frac{\partial p}{\partial Q}$ is a symmetric matrix. To verify the symmetry, set $Q = \varphi(q)$ where $\varphi = \psi^{-1}$ and $p = \varphi_q^T P$. Let G_{ij} denote the element in the i^{th} row and j^{th} column of G. Then

$$(2.3) \qquad ((\psi_Q)^T \frac{\partial p}{\partial Q})_{ij} = \sum_k \frac{\partial \psi_k}{\partial Q_i} \frac{\partial p_k}{\partial Q_j}$$

$$= \sum_{k,\ell} \frac{\partial \psi_k}{\partial Q_i} \frac{\partial \psi_\ell}{\partial Q_j} \frac{\partial}{\partial q_\ell} \sum_m (\frac{\partial \varphi}{\partial q})^T_{km} P_m$$

$$= \sum_{k,\ell} \frac{\partial^2 \Phi}{\partial q_\ell \partial q_k} \frac{\partial \psi_k}{\partial Q_i} \frac{\partial \psi_\ell}{\partial Q_j} = (\Phi'' \frac{\partial \psi}{\partial Q_i}, \frac{\partial \psi}{\partial Q_j})_{\mathbb{R}^n}$$

where $\Phi = (\varphi(q), P)_{\mathbb{R}^n} = (Q, P)_{\mathbb{R}^n}$ and Φ'' is the corresponding matrix of second partial derivatives. The symmetry of Φ'' then shows i and j can be interchanged in (2.3) and therefore the transformation is canonical. Thus (1.1) transforms to a new Hamiltonian system

$$(2.4) \qquad \dot{P} = -\hat{H}_Q, \quad \dot{Q} = \hat{H}_P$$

where

$$\hat{H}(P,Q) = H(\psi_Q(Q)^{-T} P, \psi(Q))$$

$$= K(\psi_Q(Q)^{-T} P, \psi(Q)) + V(\psi(Q)) = \hat{K}(P,Q) + \hat{V}(Q) .$$

Note in particular that $\hat{V} \in C^1(\mathbb{R}^n, \mathbb{R})$ and satisfies (V_1'). Moreover $\hat{K} \in C^1(\mathbb{R}^{2n}, \mathbb{R})$ —observe the loss of a derivative— , $\hat{K}(0, Q) = 0$, and by (K_1) for $|Q| < 1$ and $P \neq 0$,

$$(P, \hat{K}_P)_{\mathbb{R}^n} = \sum_i P_i \hat{K}_{P_i} = \sum_{i,j} P_i K_{p_j} \frac{\partial p_j}{\partial P_i}$$

$$= \sum_{i,j} P_i K_{p_j} \left(\frac{\partial \varphi}{\partial q}\right)_{ij} = (p, K_p)_{\mathbb{R}^n} > 0$$

so $\hat{K}$ satisfies (K_1'). Lastly (K_2') obtains since

$$\lim_{\alpha \to \infty} \hat{K}(\alpha P, Q) = \lim_{\alpha \to \infty} K(\alpha \psi_Q^{-T} P, \psi(Q)) > 1 - \hat{V}(Q)$$

for $|Q| < 1$ and $P \neq 0$ via (K_2).

Thus to prove Theorem 1.4, it suffices to prove Theorem 2.1. Henceforth we assume the hypotheses of Theorem 2.1 are satisfied. Let $\mathfrak{m}$ be as in Remark 1.5 with D replaced by $B_1(0)$ so $\mathfrak{m}$ is a compact C^1 manifold in $\mathbb{R}^{2n}$.

<u>Lemma 2.5</u>: Let $\overline{H} \in C^1(\mathbb{R}^{2n}, \mathbb{R})$, $\overline{H}^{-1}(1) = \mathfrak{m}$, and $\overline{H}_z \neq 0$ on $\mathfrak{m}$. If $\zeta(t)$ satisfies

$$\dot{\zeta} = \mathcal{J} \overline{H}_z(\zeta) \tag{2.6}$$

and $\zeta(0) \in \mathfrak{m}$, then there is a reparameterization $z(t)$ of ζ which satisfies (1.1). In particular if $\zeta(t)$ is periodic, so is $z(t)$.

Proof: This is Lemma 3.1 of [5]. If $H, \overline{H} \in C^2$, the proof follows immediately from the facts that (1.2) and (2.6) are Hamiltonian systems so the corresponding flows remain on $\mathfrak{m}$ and that $H_z(z) = \beta(z)\overline{H}_z(z)$ for $z \in \mathfrak{m}$ where $0 \neq \beta \in C^1(\mathfrak{m}, \mathbb{R})$. For the C^1 case, a bit more care must be taken. See [5].

On the basis of Lemma 2.5, to find a periodic solution of (1.1) on $\mathfrak{m}$, it suffices to find a suitable $\overline{H}$ and find a periodic solution for (2.6) on $\mathfrak{m}$. Such a function $\overline{H}$ will be constructed next. It possesses properties which are more amenable to the variational approach taken in §3 than does H.

By (V_1'), there are constants $\delta, \alpha > 0$ such that

$$(2.7) \qquad (V_q(q), q) \geq \alpha V(q) \geq \frac{\alpha}{2}$$

if $1 - 2\delta \leq |q| \leq 1 + 2\delta$. Moreover by (V_1') again, there is a $\mu = \mu(\delta) > 0$ such that if

$$(2.8) \qquad V(q) > 1 \quad \text{and} \quad |q| \leq 1 + 2\delta, \quad \text{then} \quad |q| > 1 - \delta .$$

Next observe that since $K(0, q) = 0$, there exists $\mu_1 = \mu_1(\delta) > 0$ such that

$$(2.9) \qquad K(p, q) < \mu/2 \ \text{ if } \ |p| < \mu_1 \ \text{ and } \ |q| \leq 1 + 2\delta .$$

Finally note that there is a constant $M_1 > 0$ such that

$$(2.10) \qquad \mathfrak{m} \subset \{z \in \mathbb{R}^{2n} \mid |z| < M_1\} .$$

We can further assume that if $|p| > M_1$ and $|q| \leq 1$,

$$(2.11) \qquad \min(K(p,q), |p|^2) > 1 .$$

For $a, b \in \mathbb{R}$ and $a < b$, let $\chi(s; a, b) \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(s; a, b) = 1$ if $s \leq a$; $= 0$ if $s \geq b$; and $\frac{d\chi}{ds} < 0$ if $a < s < b$. Set

$$(2.12) \qquad \begin{cases} \chi_1(p) = 1 - \chi(|p|; \frac{\mu_1}{2}, \mu_1) \\ \chi_2(q) = \chi(|q|; 1 - 2\delta, 1 - \delta) \\ \chi_3(q) = \chi(|q|; 1, 1 + \delta) \\ \chi_4(p) = \chi(|p|; M_1+1, M_1+2) \end{cases}$$

Now we define functions $\overline{V}$, $\overline{K}$, $\overline{H}$ as follows

$$(2.13) \qquad \begin{cases} \overline{V}(p,q) = \chi_1(p)\chi_2(q)V(q) + (1-\chi_2(q))\chi_3(q)V(q) + \rho_1(1-\chi_3(q))|q|^2 \\ \overline{K}(p,q) = \chi_4(p)\chi_3(q)K(p,q) + \rho_2(1-\chi_4(p))|p|^2 \\ \overline{H}(p,q) = \overline{K}(p,q) + \overline{V}(p,q) \end{cases}$$

where $\rho_1, \rho_2 > 0$ are free for the moment. The functions $\chi_2 - \chi_4$ insure that $\overline{H}(z)$ grows like $|z|^2$ for large $|z|$ and the χ_1 term forces $\overline{V}(p,q) = 0$ if $|p| < \mu_1/2$ and $|q| < 1 - 2\delta$.

<u>Lemma 2.14</u>: For ρ_1, ρ_2 sufficiently large,

1° $(p, \overline{H}_p(z))_{\mathbb{R}^n} \geq 0$ with strict inequality if $p \neq 0$ and $|q| < 1 + \delta$.

2° $(p, \overline{H}_p(z))_{\mathbb{R}^n} \geq 2\rho_2 |p|^2$ if $|p| \geq M_1 + 2$.

3° If $p = 0$, $(q, \overline{H}_q(0,q))_{\mathbb{R}^n} = (q, \overline{V}_q(0,q))_{\mathbb{R}^n} \geq 0$ with strict inequality if $|q| \geq 1 - \delta$.

Proof:

$$(2.15) \qquad (p, \overline{H}_p(z))_{\mathbb{R}^n} = \chi_4(p)\chi_3(q)(p, K_p(z))_{\mathbb{R}^n} + 2\rho_2(1-\chi_4(p))|p|^2$$
$$+ (\chi_3(q)K(z) - \rho_2|p|^2)(p, \chi_{4p})_{\mathbb{R}^n} + \chi_2(q)V(q)(p, \chi_{1p})_{\mathbb{R}^n}$$

from which 1° and 2° follow via (2.12), (V_1'), and (K_1') provided that ρ_2 is sufficiently large. To prove 3°, note that for $p = 0$,

$$(2.16) \qquad (p, \overline{H}_q(0,q))_{\mathbb{R}^n} = (q, \overline{V}_q(0,q))_{\mathbb{R}^n} = (1-\chi_2(q))(q, V_q(q))_{\mathbb{R}^n}$$
$$+ ((1-\chi_2(q))V(q) - \rho_1|q|^2)(q, \chi_{3q})_{\mathbb{R}^n} + 2\rho_1(1-\chi_3(q))|q|^2$$

so 3° follows as above if ρ_1 is sufficiently large.

Henceforth we assume ρ_1, ρ_2 are large enough so that the inequalities in Lemma 2.14 are valid.

<u>Lemma 2.17</u>: $\overline{H}^{-1}(1) = \mathbb{M}$.

Proof: Suppose first that $|p| \geq \mu_1$. Then $\chi_1(p) = 1$. If in addition $\overline{H}(z) = 1$, then 3° of Lemma 2.14 and the form of $\overline{H}$ implies that $|q| \leq 1$. Hence $\chi_3(q) = 1$ and

$$(2.18) \qquad 1 = \chi_4(p)K(z) + \rho_2(1-\chi_4(p))|p|^2 + V(q) .$$

We can assume $\rho_2 \geq 1$ so by (2.11), (2.18) cannot be satisfied if $|p| > M_1$. Hence $|p| \leq M_1$, $\chi_4(p) = 1$, and $H(z) = 1$. Since $|q| \leq 1$, it follows that $z = (p,q) \in \mathfrak{m}$. Conversely if $|p| \geq \mu_1$ and $z \in \mathfrak{m}$, then $\chi_4(p) = 1 = \chi_3(q)$. Therefore $\overline{H}(z) = H(z) = 1$.

Next suppose that $|p| < \mu_1$. Hence $\chi_4(p) = 1$. If $\overline{H}(z) = 1$, $|q| \leq 1$ so $\chi_3(q) = 1$ and

$$(2.19) \qquad 1 = K(z) + (\chi_1(p)\chi_2(q) + (1-\chi_2(q)))V(q) .$$

By (2.9),

$$V(q) > \frac{1 - \mu/2}{\chi_1(p)\chi_2(q) + (1-\chi_2(q))} \geq 1 - \mu \quad .$$

Therefore $|q| > 1 - \delta$ by (2.8) and $\chi_2(q) = 0$. Hence $H(z) = 1$ and $z \in \mathfrak{m}$ as above. Conversely if $|p| < \mu_1$ and $z \in \mathfrak{m}$, $\chi_4(p) = 1 = \chi_3(q)$ and

$$\overline{H}(z) = K(z) + (\chi_1(p)\chi_2(q) + (1-\chi_2(q)))V(q) \quad .$$

As above $K(z) < \mu/2$ implies $\chi_2(q) = 0$ and $\overline{H}(z) = H(z) = 1$.

Lemmas 2.5 and 2.17 reduce the proof of Theorem 2.1 to finding a periodic solution of (2.6) on $\mathbb{M}$ with $\overline{H}$ defined by (2.13). We shall determine such a solution but before doing so it is convenient to make one further modification to $\overline{H}$. Set

$$\overline{H}_\varepsilon(z) = \overline{H}(z) + \varepsilon|p|^2 . \tag{2.20}$$

Note that

$$(p, \overline{H}_{\varepsilon p}(z))_{\mathbb{R}^n} \geq 2\varepsilon|p|^2 \tag{2.21}$$

for all $z \in \mathbb{R}^{2n}$. Our goal now is to find a periodic solution of

$$\dot{z} = \mathcal{J}\overline{H}_{\varepsilon z} \tag{2.22}$$

on $\overline{H}_\varepsilon^{-1}(1)$ for all small $\varepsilon > 0$ and let $\varepsilon \to 0$ to get a periodic solution of (2.6) on $\mathbb{M}$.

§3. The existence proof

Suppose $z(t)$ is a periodic solution of (2.22). Since its period T is a priori unknown, it is convenient to make the change of time variable $t \to \frac{2\pi}{T} t \equiv \lambda^{-1} t$ so that (2.22) becomes

$$\dot{z} = \lambda \mathcal{J} \overline{H}_{\varepsilon z} \tag{3.1}$$

with z now a 2π periodic function. The parameter λ must be determined in the course of the proof.

Set $E = (W^{1,2}(S^1))^{2n}$ and for $z \in E$, let

$$\Psi(z) = \frac{1}{2\pi} \int_0^{2\pi} \overline{H}_\varepsilon(z)\, dt \ .$$

Define

$$S = \{z \in E \mid \Psi(z) = 1\} \ .$$

As was noted in the introduction, any critical point of

$$A(z) = \int_0^{2\pi} (p, \dot{q})_{\mathbb{R}^n}\, dt$$

on S with a nonzero Lagrange multiplier provides a solution of (3.1) on $H_\varepsilon^{-1}(1)$. Since we know of no direct method to find such critical points, we begin with an approximation argument.

Let $e_1, \cdots, e_{2n}$ denote the usual orthonormal basis in $\mathbb{R}^{2n}$ and set

$$E_m = \text{span}\{(\sin jt)e_k, (\cos jt)e_k \mid 0 \le j \le m, 1 \le k \le 2n\}$$

and $S_m = S \cap E_m$.

<u>Lemma 3.2</u>: S_m is a compact C^1 manifold which bounds a neighborhood of 0 in E_m.

Proof: For $z, \zeta \in E_m$, if $\Psi'(z)\zeta$ denotes the Frechet derivative of Ψ at z acting on ζ,

$$\Psi'(z)\zeta = \frac{1}{2\pi} \int_0^{2\pi} (\overline{H}_{\varepsilon z}(z), \zeta)_{\mathbb{R}^{2n}} \, dt \, . \tag{3.3}$$

Hence by (2.21),

$$\Psi'(z)(p, 0) \ge \frac{\varepsilon}{\pi} \|p\|_{L^2}^2 \, . \tag{3.4}$$

If $p = 0$, by $3°$ of Lemma 2.14,

$$\Psi'(z)(0, q) > 0 \tag{3.5}$$

unless $\|q(t)\|_{L^\infty} < 1 - \delta$. But then

$$\Psi(0, q) = \frac{1}{2\pi} \int_0^{2\pi} (1 - \chi_2(q)) V(q) \, dt < 1$$

so $(0,q) \notin S_m$. Hence $\Psi'(z) \neq 0$ for $z \in S_m$ and by the implicit function theorem, S_m is a C^1 manifold in E_m.

Next observe that if $z \in E_m$ and $\|z\|_E = 1$,

$$\Psi(rz) \geq \frac{r^2}{2\pi} \int_0^{2\pi} [\rho_2(1-\chi_4(rp))|p|^2 + \rho_1(1-\chi_3(rq))|q|^2]\, dt \to \infty \tag{3.6}$$

as $r \to \infty$. Hence S_m is compact. Lastly observe that for $(0,q) \in E_m$ with $\|q\|_{L^2} = 1$, by (3.5) - (3.6) there is a unique $r = r(q)$ such that $\Psi(0,rq) = 1$. Since $\Psi(p,q) \geq \Psi(0,q)$, (3.4) then shows if $\|q\|_{L^2} < r(q/\|q\|_{L^2})$ (if $q \neq 0$) and $\|p\|_{L^2} = 1$, there is a unique $s = s(p,q) > 0$ such that $\Psi(sp,q) = 1$. It follows that S_m bounds a *neighborhood of* 0 *in* E_m *which for fixed* q *is star-shaped with respect to* p. The lemma is proved.

Next we exploit some invariance properties inherent in our spaces and operators. Let E_m^+, E_m^-, E^0 denote respectively the subspaces of E_m on which A is positive definite, negative definite, and null. It is easy to represent these spaces explicitly, namely

$$E^0 = \text{span}\{e_k \mid 1 \leq k \leq 2n\},$$

$$E_m^+ = \text{span}\{(\sin jt)e_k - (\cos jt)e_{k+n}, (\cos jt)e_k + (\sin jt)e_{k+n} \mid 1 \leq j \leq m, \; 1 \leq k \leq n\}$$

$$E_m^- = \text{span}\{(\sin jt)e_k + (\cos jt)e_{k+n}, (\cos jt)e_k - (\sin jt)e_{k+n} \mid 1 \leq j \leq m, \; 1 \leq k \leq n\}.$$

These subspaces are orthogonal under the L^2 inner product and are invariant under the family of mappings $z(t) \to z(t+\tau)$ for all $\tau \in [0, 2\pi]$.

This family of mappings induces an S^1 action on E_m (see [8] or [3]). A cohomological index theory for compact Lie group actions developed in [8] can be applied to our setup here. Let $\mathcal{E}_m$ denote the family of subsets of $E_m \setminus \{0\}$ which are invariant under the above S^1 action, i.e. $B \subset \mathcal{E}_m$ implies $z(t+\tau) \in B$ for all $\tau \in [0, 2\pi]$ whenever $z(t) \in B$. A mapping $f : E_m \to E_m$ will be called equivariant if $f : \mathcal{E}_m \to \mathcal{E}_m$. Similarly a mapping $g : E_m \to \mathbf{R}$ will be called equivariant if $g(z(t+\tau)) = g(z(t))$ for all $\tau \in [0, 2\pi]$ and $z \in E_m$. Note that A and Ψ are equivariant mappings.

Lemma 3.7: There is an index theory, i.e. a mapping $i : \mathcal{E}_m \to \mathbb{N} \cup \{\infty\}$ which possesses the following properties: If $B, \hat{B} \in \mathcal{E}_m$,

1° $i(B) < \infty$ if and only if $B \cap E^0 = \emptyset$.

2° If there is an $f \in C(B, \hat{B})$ with f equivariant, then $i(B) \le i(\hat{B})$.

3° $i(B \cup \hat{B}) \le i(B) + i(\hat{B})$.

4° If F is an invariant subspace of $E_m^+ \oplus E_m^-$ and $\mathcal{S}$ is the boundary of a bounded open invariant neighborhood of 0 in F, then $i(\mathcal{S}) = \frac{1}{2} \dim F$.

5° If $B \in \mathcal{E}_m$ with $i(B) \ge mn$, F is an invariant subspace of E_m containing E^0 and having $\dim F \ge 2mn + 2n + 2$, then $B \cap F \ne \emptyset$.

Proof: The definition of the index and proofs of 1° - 4° as well as other properties of index can be found in §6-7 of [8] and 5° is Lemma 1.24 of [3].

An immediate consequence of Lemma 3.7 is

Lemma 3.8: $i(S_m \cap E_m^-) = mn$.

Proof: Since $\dim E_m^- = 2mn$ and E_m^- is an invariant subspace of $E_m^+ \oplus E_m^-$, Lemma 3.2 implies $S_m \cap E_m^-$ is the boundary of an open neighborhood Ω of 0 in E_m^-. The equivariance of Ψ implies $\Omega \in \mathcal{E}_m$. Hence the result follows from 4° of Lemma 3.7.

At this point it is convenient to further assume that H (and therefore $\overline{H}_\varepsilon) \in C^2(\mathbb{R}^{2n}, \mathbb{R})$. We will return to the C^1 case later. Since A is an equivariant mapping defined on $S_m \in \mathcal{E}_m$, there is a standard method for trying to find critical values of $A|_{S_m}$. Define

$$\gamma_{j,m} = \inf_{\substack{B \subset S_m \\ i(B) \geq j}} \max_{z \in B} A(z) \qquad 1 \leq j \leq mn \tag{3.9}$$

Lemma 3.10: $\gamma_{j,m}$ is a negative critical value of $A|_{S_m}$, $1 \leq j \leq mn$.

Proof: Since $i(S_m \cap E_m^-) = mn$,

$$\gamma_{1,m} \leq \cdots \leq \gamma_{mn,m} \leq \max_{S_m \cap E_m^-} A < 0 .$$

Since $H \in C^2$, the remainder of the proof is standard. See e.g. Lemma 1.16 in [3] .

Not all of the critical values obtained in Lemma 3.10 are of use to us. E.g. since $A|_S$ is not bounded from below, $\gamma_{1,m} \to -\infty$ as $m \to \infty$. Accordingly we focus our attention on $c_m \equiv \gamma_{mn,m}$. Heuristically c_m is the largest negative critical value that $A|_{S_m}$ possesses. Let z_m be a critical point corresponding to c_m, i.e. $A(z_m) = c_m$ and

$$(3.11) \qquad A'(z_m)\zeta - \lambda_m \Psi'(z_m)\zeta = 0$$

for all $\zeta \in E_m$. To obtain a solution of (3.1), we will find upper and lower bounds for c_m, for λ_m, and finally bounds for $\|z_m\|_E$. These estimates will be determined in the following series of lemmas.

<u>Lemma 3.12</u>: There are constants $\alpha_1, \alpha_2 < 0$ and independent of m such that

$$(3.13) \qquad \alpha_1 \le c_m \le \alpha_2 .$$

Moreover there is an $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0]$, α_1 and α_2 are independent of ε.

Proof: Since $\overline{H}_\varepsilon(0) = 0$, (2.13) shows there is a constant $M_2 > 0$ and independent of ε such that

(3.14) $$\overline{H}_\varepsilon(z) \le \frac{1}{2} + (M_2 + \varepsilon)|z|^2$$

for all $z \in \mathbf{R}^{2n}$. Choosing $z = z(t) \in S$, (3.14) yields

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \overline{H}_\varepsilon(z)dt \le \frac{1}{2} + \frac{M_2+\varepsilon}{2\pi}\|z\|^2_{L^2}.$$

Hence

(3.15) $$\|z\|_{L^2} \ge \left(\frac{\pi}{M_2+\varepsilon}\right)^{\frac{1}{2}} \ge \left(\frac{\pi}{2M_2}\right)^{\frac{1}{2}} \equiv M_3$$

for all $z \in S$ provided that $0 \le \varepsilon \le \varepsilon_0 \le M_2$ which choice we make. Similarly (2.13) shows if $\rho_3 = \min(\rho_1, \rho_2)$, *there is a constant* $\alpha_3 > 0$ and independent of ε such that

(3.16) $$\overline{H}_\varepsilon(z) \ge \frac{\rho_3}{2}|z|^2 - \alpha_3$$

for all $z \in \mathbf{R}^{2n}$. Again choosing $z = z(t) \in S$ in (3.16) gives

(3.17) $$\|z\|_{L^2} \le \left(\frac{4\pi(1+\alpha_3)}{\rho_3}\right)^{1/2} \equiv M_4 .$$

Thus we have upper and lower bounds for $z \in S$ independent of $\varepsilon \in (0, \varepsilon_0]$. To get the bounds for c_m, note first that the form of A and E_m^- imply

(3.18) $$c_m \le \max_{z \in S_m \cap E_m^-} A(z) \le \max\{A(z) \mid z \in E_m^-, \|z\|_{L^2} = M_3\}$$
$$= -\pi M_3^2 \equiv \alpha_2 .$$

To get the lower bound, let

$$F = E_m^+ \oplus E^0 \oplus \operatorname{span}\{(\sin t)e_1 + (\cos t)e_{k+1}, (\cos t)e_1 - (\sin t)e_{k+1}\} .$$

Then $\dim F = 2mn + 2n + 2$, $F \supset E^0$, and F is an invariant subspace of E_m. Thus by 5° of Lemma 3.7, if $B \in S_m$ and $i(B) \geq mn$, then $B \cap F \neq \emptyset$. Choosing any $\tilde{z} \in B \cap F$ shows

$$\min_{F \cap S_m} A \leq A(\tilde{z}) \leq \max_B A . \tag{3.19}$$

Since (3.19) holds for all $B \subset S$ in satisfying $i(B) \geq mn$, (3.17) and (3.19) yield

$$\begin{aligned} c_m \geq \min_{F \cap S_m} A &\geq \min\{A(z) \mid z \in F, \|z\|_{L^2} = M_4\} \\ &= -\pi M_4^2 \equiv \alpha_1 . \end{aligned} \tag{3.20}$$

Remark 3.21: We now show how to treat the case in which H and therefore $\overline{H}_\varepsilon$ is merely assumed to belong to $C^1(\mathbb{R}^{2n}, \mathbb{R})$. Let $V_k(q), K_k(z)$ be sequences of functions which converge uniformly in the C^1 norm to V and K respectively on $\{(p,q) \in \mathbb{R}^{2n} \mid |p| \leq M_2 + 2, |q| \leq 1 + \delta\}$. Replacing V and K by V_k, K_k respectively in (2.13) and (2.20) defines functions $\overline{H}_k(z)$, $\overline{H}_{k\varepsilon}(z)$ where $\overline{H}_{k\varepsilon}(z) \to \overline{H}_k(z)$ uniformly in the C^1 norm on $\mathbb{R}^{2n}$ as $k \to \infty$. Set

$$\Psi_k(z) = \frac{1}{2\pi} \int_0^{2\pi} \bar{H}_{k\varepsilon}(z)\, dt \ .$$

Then Ψ_k is equivariant on E_m and $\Psi_k \to \Psi$ in the C^1 norm uniformly on E_m. Moreover (3.4) - (3.6) show that for large k, $\Psi_k^{-1}(1)$ is a C^2 compact manifold in E_m which is the boundary of a bounded neighborhood of 0. By the C^2 case, for each such k,

$$\gamma^k_{j,m} \equiv \inf_{\substack{B \subset \Psi_k^{-1}(1) \\ i(B) \geq j}} \ \max_{z \in B} A(z) \ , \qquad 1 \leq j \leq mn$$

is a negative critical value of $A|_{\Psi_k^{-1}(1) \cap E_m}$. We will only study $c^k_m \equiv \gamma^k_{mn,m}$. Let z^k_m be a corresponding critical point. Since these critical points are uniformly bounded in the finite dimensional space E_m and $\Psi_k \to \Psi$ uniformly in C^1 on E_m, a subsequence of z^k_m converges to a critical point z_m of A on S_m. Define $c_m = A(z_m)$ so by definition c_m is a critical value of A on S_m. An inspection of the proof of Lemma 3.12 shows the estimates (3.14) - (3.17) with S replaced where appropriate by $\Psi_k^{-1}(1)$ can be assumed independent of k as well as m and ε. Hence the bounds (3.18) and (3.20) hold for c^k_m independently of m, ε, and large k. Letting $k \to \infty$, we see (3.13) holds for c_m.

For the remainder of the proof of Theorem 2.1, we assume we are in the $H \in C^1$ case. The next lemma provides ε dependent bounds for λ_m.

<u>Lemma 3.22</u>: There are constants $\varepsilon_1, \varepsilon_2 > 0$ and independent of m such that

$$\beta_1 \le \lambda_m \le \beta_2 \ . \tag{3.23}$$

Proof: From (3.11) with $z_m = (p_m, q_m)$ and $\zeta = (p_m, 0)$, we find

$$c_m = \int_0^{2\pi} (p_m, \dot{q}_m)_{\mathbb{R}^n}\, dt = \lambda_m \int_0^{2\pi} (p_m, \overline{H}_{\varepsilon p}(z_m))_{\mathbb{R}^n}\, dt \ . \tag{3.24}$$

Since $c_m < 0$, (2.21) shows $\lambda_m < 0$. Moreover by (3.24), (2.15), and (3.17)

$$\frac{c_m}{\lambda_m} \le M_5 + 2(\rho_2 + \varepsilon) M_4^2$$

which together with (3.13) gives the upper bound for λ_m.

A lower bound for λ_m is more difficult to obtain. In fact we will only obtain an indirect estimate. Thus suppose $\lambda_m \to -\infty$ as $m \to \infty$ along some subsequence. Since by (2.21),

$$\frac{c_m}{\lambda_m} = \int_0^{2\pi} (p_m, \overline{H}_{\varepsilon p}(z_m))_{\mathbb{R}^n}\, dt \ge 2\varepsilon \| p_m \|_{L^2}^2 \ ,$$

the integrals

$$\int_0^{2\pi} (p_m, \overline{H}_{\varepsilon p}(z_m))_{\mathbb{R}^n}\, dt \tag{3.25}$$

tend to 0 as $m \to \infty$ along some subsequence. But (3.25) consists of five non negative terms (see 2.25), so each term tends to 0 as $m \to \infty$. In particular $\varepsilon \| p_m \|_{L^2} \to 0$ and if

$$\tau_m(\sigma) = \text{measure}\,\{t \in [0, 2\pi] \mid p_m(t) \geq \sigma\}\,,$$

then for all $\sigma > 0$, $\tau_m(\sigma) \to 0$ as $m \to \infty$ along our subsequence. Since

$$(3.26) \qquad 1 = \Psi(z_m) = \frac{1}{2\pi}\int_0^{2\pi} [\chi_4(p_m)\chi_3(q_m)K(z_m) + $$
$$+ \rho_2(1-\chi_4(p_m))\,|p_m|^2 + \varepsilon|p_m|^2 + \chi_1(p_m)\chi_2(q_m)V(q_m) +$$
$$+ (1-\chi_2(q_m))\chi_3(q_m)V(q_m) + \rho_1(1-\chi_3(q_m))|q_m|^2]\,dt\,,$$

it easily follows that the first four terms on the right hand side of (3.26) tend to 0 as $m \to \infty$. Moreover choosing $\zeta = (0, q_m)$ in (3.11) yields

$$(3.27) \qquad \frac{c_m}{\lambda_m} = \int_0^{2\pi} (q_m, \overline{H}_{\varepsilon q}(z_m))_{\mathbb{R}^n}\,dt = \int_0^{2\pi} [(q_m, \chi_4(p_m)\chi_3(q_m)K_q(z_m))_{\mathbb{R}^n}$$
$$+ (q_m, \chi_4(p_m)K(z_m)\chi_{3q}(q_m))_{\mathbb{R}^n} + (q_m, \chi_1(p_m)\chi_2(q_m)V_q(q_m))_{\mathbb{R}^n}$$
$$+ (q_m, \chi_1(p_m)V(q_m)\chi_{2q}(q_m))_{\mathbb{R}^n} + (q_m, (1-\chi_2(q_m))\chi_3(q_m)V_q(q_m))_{\mathbb{R}^n}$$
$$- (q_m, \chi_3(q_m)V(q_m)\chi_{2q}(q_m))_{\mathbb{R}^n} + (q_m, ((1-\chi_2(q_m))V(q_m) - \rho_1|q_m|^2)\chi_{3q}(q_m))_{\mathbb{R}^n}$$
$$+ 2\rho_1(1-\chi_3(q_m))|q_m|^2]\,dt\,.$$

Since $K(0, q) = 0$, $K_q(0, q) = 0$ and the first two terms on the right hand side of (3.27) go to 0 as $m \to \infty$ along our subsequence. Likewise as in (3.26) the next two terms in (3.27) tend to 0 as $m \to \infty$. The remaining four terms are non negative and since the left hand side of (3.27) tends to 0, each of these terms also goes to 0 as $m \to \infty$. In particular

$$\text{(3.28)} \qquad \int_0^{2\pi} (1-\chi_3(q_m))|q_m|^2\, dt \to 0$$

and

$$\text{(3.29)} \qquad \int_0^{2\pi} (1-\chi_2(q_m))\chi_3(q_m)(q_m, V_q(q_m))_{\mathbb{R}^n}\, dt \to 0$$

as $m \to \infty$. On examining (3.26) again and using (3.28), we conclude that

$$\text{(3.30)} \qquad \frac{1}{2\pi} \int_0^{2\pi} (1-\chi_2(q_m))\chi_3(q_m)V(q_m)\, dt \to 1$$

as $m \to \infty$. But $(1-\chi_2(s))\chi_3(s)$ is nonzero only for $1 - 2\delta < s < 1 + \delta$. Hence by (2.7) and (3.30),

$$\text{(3.31)} \qquad \frac{1}{2\pi} \int_0^{2\pi} (1-\chi_2(q_m))\chi_3(q_m)(q_m, V_q(q_m))_{\mathbb{R}^n}\, dt \geq$$

$$\geq \frac{\alpha}{2\pi} \int_0^{2\pi} (1-\chi_2(q_m))\chi_3(q_m)V(q_m)\, dt \to \frac{\alpha}{2\pi}$$

as $m \to \infty$ along our subsequence, contrary to (3.29). Consequently there must exist ρ_1 as in the statement of the lemma.

One final estimate is required:

<u>Lemma 3.32</u>: There is a constant $M_\varepsilon > 0$ and independent of m such that $\|z_m\|_E \leq M_\varepsilon$.

Proof: Using (3.11) with $\zeta = (\dot{q}_m, -\dot{p}_m)$ and the Schwarz inequality yields

(3.33) $$\|\dot{z}_m\|_{L^2} \le |\lambda_m| \, \|\overline{H}_{\varepsilon z}(z_m)\|_{L^2} .$$

Since $\overline{H}_\varepsilon(z)$ grows at most linearly in z, the bound for $\|z_m\|_E$ now follows from (3.17), (3.33), and (3.23).

Combining the estimates given in the above lemmas now gives an existence result.

Lemma 3.34: There exists a solution $(\lambda_\varepsilon, z_\varepsilon)$ of (3.1) with $z_\varepsilon \in C^1(S^1, \mathbb{R}^{2n})$ and $\overline{H}_\varepsilon(z_\varepsilon(t)) \equiv 1$.

Proof: Lemmas 3.22 and 3.32 and the Sobolev imbedding theorem imply that along some subsequence $z_m \to z_\varepsilon$ weakly in E and strongly in L^∞ and $\lambda_m \to \lambda_\varepsilon < 0$ where $(\lambda_\varepsilon, z_\varepsilon)$ satisfies (3.11) for all $\zeta \in \bigcup_{m \in \mathbb{N}} E_m$. Hence $z_\varepsilon \in S$ and is a weak solution of (3.1) with $\lambda = \lambda_\varepsilon$. It readily follows that z_ε satisfies (3.1) pointwise a.e. But since $\overline{H}_{\varepsilon z}(z_\varepsilon)$ is continuous, z_ε must be continuously differentiable. Hence z_ε is a classical solution of (3.1). Lastly $\overline{H}_\varepsilon(z_\varepsilon) \equiv$ constant since (3.1) is a Hamiltonian system. Therefore $z_\varepsilon \in S$ implies that $\overline{H}_\varepsilon(z_\varepsilon) = 1$.

It remains to let $\varepsilon \to 0$ and obtain a solution of

$$\dot{z} = \lambda \mathcal{J} \overline{H}_z$$

on $\mathbb{M}$ which via previous remarks will then complete the proof of Theorem 2.1. We begin with ε independent a priori bounds for (λ, z) for 2π periodic solutions of (3.1) on $\mathbb{M}$. Recall that ε_0 was defined in Lemma 3.12, μ in (2.8), and M_1 in (2.10).

Lemma 3.35: Let (λ, z) be a solution of (3.1) with $z \in C^1(S^1, \mathbb{R}^{2n})$, $A(z) = c < 0$, and $\overline{H}_\varepsilon(z_\varepsilon) = 1$. If $\varepsilon_1 = \min(\varepsilon_0, {}^{\mu}/_{2M_1^2})$ and $\varepsilon \in [0, \varepsilon_1]$, then there are constants $M_7 - M_9 > 0$ and independent of ε such that

$$
(3.36) \qquad \begin{cases} \dfrac{c}{M_7} \le \lambda \le \dfrac{c}{M_8} \\ \\ \|z\|_{C^1} \le M_9 \end{cases}
$$

Proof: As earlier if $z = (p, q)$,

$$
(3.37) \qquad \frac{c}{\lambda} = \int_0^{2\pi} (p, \overline{H}_{\varepsilon p}(z))_{\mathbb{R}^n} \, dt .
$$

If $\overline{H}_\varepsilon(z) = 1$, $|q| \le 1$ via e.g. Lemma 2.17 and the definition of $\overline{H}_\varepsilon$. Moreover the form of $\overline{H}_\varepsilon$ implies $|p| < M_1$. Hence the integrand on the right hand side of (3.37) is bounded from above by a constant independent of ε from which the upper bound for λ follows.

To obtain the lower bound, let

$$
\tau(\sigma) = \text{measure}\,\{t \in [0, 2\pi] \,\big|\, |p(t)| \ge \sigma\} .
$$

Then

$$(3.38) \qquad \frac{c}{\lambda} = \int_0^{2\pi} (q, \overline{H}_{\varepsilon q}(z))_{\mathbf{R}^n} \, dt = I_1 + I_2$$

where I_1 (resp. I_2) denotes the integral over the set in which $|p(t)| < \sigma$ (resp. $|p(t)| \geq \sigma$) and σ is free for now. The bounds for $|p|$, $|q|$ on $H_\varepsilon^{-1}(1)$ imply that there is a constant $M > 0$ such that

$$(3.39) \qquad |(q, \overline{H}_{\varepsilon q}(z))_{\mathbf{R}^n}| \leq M$$

for all $\varepsilon \in [0, \varepsilon_0]$ and $z = (p, q) \in H_\varepsilon^{-1}(1)$. Therefore (3.38) - (3.39) imply that

$$(3.40) \qquad \frac{c}{\lambda} \geq I_1 - \tau(\sigma) M \;.$$

Choosing

$$(3.41) \qquad \sigma < \mu_1 / 2 \;,$$

$\chi_1(p) = 0$ and

$$(3.42) \qquad \overline{H}_\varepsilon(z) = 1 = K(p, q) + (1 - \chi_2(q)) V(q) + \varepsilon |p|^2 \;.$$

Hence by (2.9), our choice of ε, (3.42), and (2.8), $|q| > 1 - \delta$. Consequently $\chi_2(q) = 0$ and the integrand in the I_1 term is

$$(q, K_q(z))_{\mathbf{R}^n} + (q, V_q(q))_{\mathbf{R}^n} \ .$$

Set

$$\omega(\sigma) = \max \{ ((q, K_q(z))_{\mathbf{R}^n} \mid \Big| \ |q| \le 1, \ |p| \le \sigma \} \ .$$

Since $K_q(0, q) = 0$, $\omega(\sigma) \to 0$ as $\sigma \to 0$. Thus by (2.8) and (2.7) for $\sigma < {}^{\mu}1/_2$,

$$I_1 \ge (2\pi - \tau(\sigma)) \left(\frac{\alpha}{2} - \omega(\sigma)\right) .$$

Further choose σ such that

(3.43) $$\omega(\sigma) \le \frac{\alpha}{4} \ .$$

Then

$$I_1 \ge (2\pi - \tau(\sigma)) \frac{\alpha}{4}$$

and by (3.40)

$$\frac{c}{\lambda} \ge \frac{\alpha\pi}{2} - \tau(\sigma)(M + \frac{\alpha}{4}) \ .$$

Thus if

$$(3.44) \qquad \tau(\sigma) < \xi \equiv \frac{\alpha\pi}{4M+\alpha} \quad ,$$

$$(3.45) \qquad \frac{c}{\lambda} \geq \frac{\alpha\pi}{4} \quad .$$

If, on the other hand, $\tau(\sigma) \geq \xi$, by (3.37) and (3.42),

$$(3.46) \qquad \frac{c}{\lambda} \geq \int_0^{2\pi} (p, K_p(z))_{\mathbb{R}^n}\, dt \geq \tau(\sigma)\,\gamma(\sigma) \geq \xi\,\gamma(\sigma)$$

where

$$\gamma(\sigma) = \min\{(p, K_p(z))_{\mathbb{R}^n} \mid \sigma \leq |p| \leq M_1,\ |q| \leq 1\} > 0 \ .$$

In any event, (3.45) - (3.46) show

$$(3.47) \qquad \frac{c}{\lambda} \geq \min\left(\frac{\alpha\pi}{4}, \xi\,\gamma(\sigma)\right) \equiv M_7$$

where σ is now fixed and satisfies (3.41) and (3.43). Thus (3.47) provides the desired lower bound on λ.

Finally the differential equation (3.1) coupled with our upper and lower bounds for λ and the pointwise bounds for $z(t)$ yield the bound for $\|z\|_{C^1}$.

Completion of proof of Theorem 2.1: Let $\varepsilon \to 0$. The bounds (3.36) and (3.13) which hold for $c_\varepsilon = A(z_\varepsilon)$ together with the differential equation

(3.1) show a subsequence of $(\lambda_\varepsilon, z_\varepsilon)$ converge to a classical solution (λ, z) of

(3.48) $$\dot{z} = \lambda \mathcal{J} \overline{H}_z$$

with z 2π periodic and $z \in \mathfrak{m}$.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The existence of periodic solutions having prescribed energy for a Hamiltonian system of ordinary differential equations is studied. It is shown in particular that if the Hamiltonian is of classical type, i.e., $H(p,q)=K(p,q) + V(q)$ where $p,q \in \mathbb{R}^n$, and K and V satisfy $K(0,q)=0$, $p \cdot K_p(p,q) > 0$, $K(p,q) \to \infty$ as $|p| \to \infty$, $D = \{q \in \mathbb{R}^n \mid 0 < V(q) < 1\}$ is diffeomorphic to the unit ball in $\mathbb{R}^n$ and $V_q \neq 0$ on ∂D, then Hamilton's equations (*) $\dot{p} = -H_q$, $\dot{q} = H_p$, have a periodic solution on $H^{-1}(1)$.

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